Now it's Time for…

- **Recurrence**
- **Relations**

Recurrence Relations

- A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n is terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with
$n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a **solution** of a recurrence relation if it terms satisfy the recurrence relation.

Recurrence Relations

•In other words, a recurrence relation is like a recursively defined sequence, but **without specifying any initial values (initial conditions)**.

•Therefore, the same recurrence relation can have (and usually has) **multiple solutions**.

•If **both** the initial conditions and the recurrence relation are specified, then the sequence is **uniquely** determined.

Recurrence Relations

•**Example:**

Consider the recurrence relation

$a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

•Is the sequence $\{a_n\}$ with $a_n=3n$ a solution of this recurrence relation?

•For $n \geq 2$ we see that

$2a_{n-1} - a_{n-2} = 2(3(n - 1)) - 3(n - 2) = 3n = a_n$.

•Therefore, $\{a_n\}$ with $a_n=3n$ is a solution of the recurrence relation.

Recurrence Relations

- Is the sequence $\{a_n\}$ with $a_n=5$ a solution of the same recurrence relation?
- For $n \geq 2$ we see that

$2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n$.

- Therefore, $\{a_n\}$ with $a_n=5$ is also a solution of the recurrence relation.

Modeling with Recurrence Relations

- **Example:**
- Someone deposits $10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years?

- **Solution:**
- Let P_n denote the amount in the account after n years.
- How can we determine P_n on the basis of P_{n-1}?

Modeling with Recurrence Relations

- We can derive the following **recurrence relation**:
- $P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$.
- The initial condition is $P_0 = 10{,}000$.
- Then we have:
- $P_1 = 1.05P_0$
- $P_2 = 1.05P_1 = (1.05)^2P_0$
- $P_3 = 1.05P_2 = (1.05)^3P_0$
- …
- $P_n = 1.05P_{n-1} = (1.05)^nP_0$

- We now have a **formula** to calculate P_n for any natural number n and can avoid the iteration.

Modeling with Recurrence Relations

- Let us use this formula to find P_{30} under the
- initial condition $P_0 = 10,000$:

- $P_{30} = (1.05)^{30} \cdot 10,000 = 43,219.42$

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After 30 years, the account contains $43,219.42.

Modeling with Recurrence Relations

- **Another example:**
- Let a_n denote the number of bit strings of length n that do not have two consecutive 0s ("valid strings"). Find a recurrence relation and give initial conditions for the sequence $\{a_n\}$.

- **Solution:**
- Idea: The number of valid strings equals the number of valid strings ending with a 0 plus the number of valid strings ending with a 1.

Modeling with Recurrence Relations

- Let us assume that $n \geq 3$, so that the string contains at least 3 bits.
- Let us further assume that we know the number a_{n-1} of valid strings of length $(n - 1)$.
- Then how many valid strings of length n are there, if the string ends with a 1?
- There are a_{n-1} such strings, namely the set of valid strings of length $(n - 1)$ with a 1 appended to them.
- **Note:** Whenever we append a 1 to a valid string, that string remains valid.

Modeling with Recurrence Relations

- Now we need to know: How many valid strings of length n are there, if the string ends with a **0**?
- Valid strings of length n ending with a 0 **must have a 1 as their (n – 1)st bit** (otherwise they would end with 00 and would not be valid).
- And what is the number of valid strings of length (n – 1) that end with a 1?
- We already know that there are a_{n-1} strings of length n that end with a 1.
- Therefore, there are a_{n-2} strings of length (n – 1) that end with a 1.

Modeling with Recurrence Relations

- So there are a_{n-2} valid strings of length n that end with a 0 (all valid strings of length (n – 2) with 10 appended to them).

- As we said before, the number of valid strings is the number of valid strings ending with a 0 plus the number of valid strings ending with a 1.

- That gives us the following **recurrence relation**:
- $a_n = a_{n-1} + a_{n-2}$

Modeling with Recurrence Relations

- What are the **initial conditions**?

- $a_1 = 2$ (0 and 1)
- $a_2 = 3$ (01, 10, and 11)
- $a_3 = a_2 + a_1 = 3 + 2 = 5$
- $a_4 = a_3 + a_2 = 5 + 3 = 8$
- $a_5 = a_4 + a_3 = 8 + 5 = 13$
- …

- This sequence satisfies the same recurrence relation as the **Fibonacci sequence**.
- Since $a_1 = f_3$ and $a_2 = f_4$, we have $a_n = f_{n+2}$.

Solving Recurrence Relations

- In general, we would prefer to have an **explicit formula** to compute the value of a_n rather than conducting n iterations.
- For one class of recurrence relations, we can obtain such formulas in a systematic way.
- Those are the recurrence relations that express the terms of a sequence as **linear combinations** of previous terms.